Solvay Brussels School
Economics & Management

ULB

Financial Risk Management and Governance

Correlations / Covariances.

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$$\begin{bmatrix} \sigma_a^2 & \sigma_{ab} & \sigma_{ac} \\ \sigma_{ab} & \sigma_b^2 & \cdots \\ \cdots & \cdots & \sigma_c^2 \end{bmatrix}$$

Introduction

- Correlation
  - $\rho \in [-1,1]$
  - The volatilities of two variables can be high but if they are not perfectly correlated, some compensation or offsetting of their movements can take place.
  - Fields
    - ✓ Diversification in portfolio theory
    - ✓ Diversification of exposures in risk management
    - ✓ Can enable us to find ways to mitigate a fraction of our net risk exposure therefore enhancing our investment capacity
- Copulas
  - Ways to define a correlation structure between two variables regardless of the shape of their probability distributions.
  - Useful for
    - ✓ Understanding some Basel II formulas
    - ✓ Modelling correlated defaults in loan/bond portfolios
    - ✓ Valuing credit derivatives on multi-entities
    - ✓ The calculation of economic capital

Definitions

- Correlation between two variables V_1 and V_2, is defined as

$$\rho = \frac{\mathrm{E}(V_1 V_2) - \mathrm{E}(V_1)E(V_2)}{\sigma(V_1)\sigma(V_2)}$$

- The covariance is $\mathrm{cov}(V_1 V_2) = \mathrm{E}(V_1 V_2) - \mathrm{E}(V_1)E(V_2)$

  » And therefore $\rho = \dfrac{\mathrm{cov}(V_1 V_2)}{\sigma(V_1)\sigma(V_2)}$

  $\beta = \text{slope}(x, y)$

  $\beta = \dfrac{\sigma_Y}{\sigma_X} \cdot \rho_{x,y}$

  » Although correlations are more intuitive than covariances, the latter will be our key variable in the analysis, a little bit like variances to volatilities in our last chapter.

≠ causality

Correlation vs. Dependence

$y = a + b.x$

- Two variables are independent
  - If knowledge about one of them does not affect the probability distribution of the other one, i.e.

$$f(V_2 | V_1 = x) = f(V_2)$$

where $f(\cdot)$ is the probability density function

a. Granger causality test

b. VAR: vector autoregression

$y = a + b_1 . x_{-1} + b_2 . x_{-2} + \dots$

$x = a + \beta_1 . y_{-1} + \beta_2 . y_{-2} + \dots$

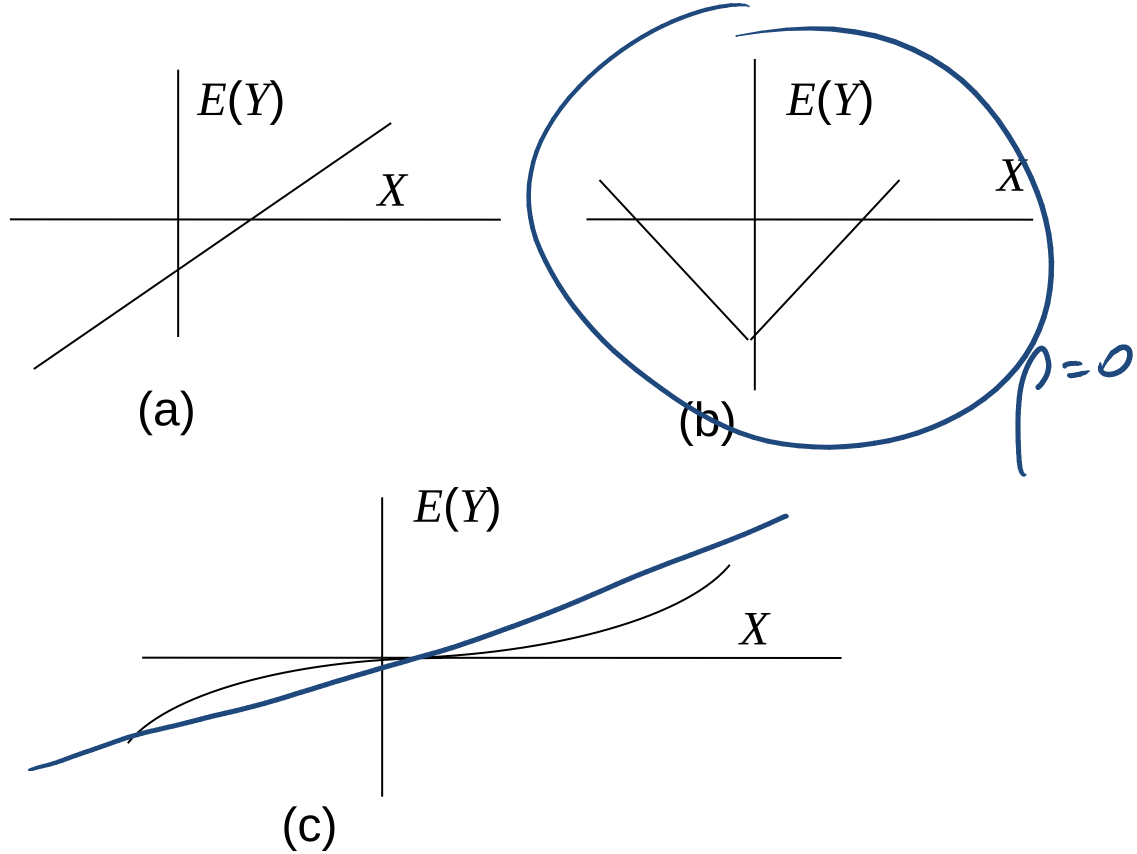
- Independence is not the same as zero correlation!
  - Suppose V_1 = –1, 0, or +1 (equally likely)
  - If V_1 = -1 or V_1 = +1 then V_2 = 1
  - If V_1 = 0 then $V_2 = 0$

  V_2 is clearly dependent on V_1 (and vice versa) but the coefficient of correlation is zero

- Correlation measures one particular type of dependence between two variables: **linear dependence**.

Other forms of dependence

$y = |x|$



$\rho = 0$

Monitoring correlation

- Similar approaches to EWMA and GARCH for variance, but for covariances, can be used to monitor the evolution of covariances.
- Define $x_i = \frac{X(i) - X(i-1)}{X(i-1)}$ and $y_i = \frac{Y(i) - Y(i-1)}{Y(i-1)}$
- The covariance on day *n* is

$$\text{cov}_n = \text{E}(x_n y_n) - \text{E}(x_n)\text{E}(y_n)$$

  - » If the expected daily return for risk managers is 0, then the simplification made for variances can also be done for covariances

$$\text{cov}_n = \text{E}(x_n y_n)$$

  - » Using equal weighting for the last *m* observations, the correlation estimate on day *n* (calculated given data up to day *n*-1) is

$$\text{cov}_n = \frac{1}{m}\sum_{i=1}^{m} x_{n-i}\ y_{n-i} \quad \text{with} \quad \text{var}_{x,n} = \frac{1}{m}\sum_{i=1}^{m} x_{n-i}\ ,\ \text{var}_{y,n} = \frac{1}{m}\sum_{i=1}^{m} y_{n-i}$$

$$\Rightarrow \rho = \frac{\text{cov}_n}{\sqrt{\text{var}_{x,n}\ \text{var}_{y,n}}}$$

Monitoring correlation (2)

- EWMA

$$\text{cov}_n = \lambda\,\text{cov}_{n-1} + (1-\lambda)x_{n-1}y_{n-1}$$

  - Example
    - ✓ λ = 0.95
    - ✓ Correlation between X and Y on day n-1: 0.6
    - ✓ Volatilities of X and Y are 1%, 2%.

    Covariance =
    - ✓ If %returns on day n of X and Y are 0.5% and 2.5%
    - ✓ $\sigma^2_{x,n}$ =
    - ✓ $\sigma^2_{y,n}$ =
    - ✓ cov_n =

    new correlation =

- GARCH(1,1)

$$\text{cov}_n = \omega + \alpha x_{n-1}y_{n-1} + \beta\,\text{cov}_{n-1} \quad \text{with long-run average covariance rate } \omega/(1-\alpha-\beta)$$

  - Some weight is given to
    - ✓ a long-run average covariance
    - ✓ the most recent covariance estimate
    - ✓ the most recent observation on covariance

Consistency condition

$$\begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix}$$ exception

- Variances and covariances for a set of variables produce a **variance-covariance matrix** (Σ)
- Not all these matrices are internally consistent, cross correlation must be overall consistent.
  » An easy condition is to ensure that $\boxed{\boldsymbol{w}^T \Sigma \boldsymbol{w} \geq 0}$ = "covariances/correlation must be coherent with one another".
  » for all $N \times 1$ vectors $\boldsymbol{w}$. The matrix is said to be *positive-semidefinite*.
- Make sure you compute variances and covariances accordingly.
  » Variances and covariances must be updated using the same method (simple, EWMA, GARCH(1,1), etc...)
  » Making changes to var-covar matrices is dangerous if we manipulate a large number of variables, because it is not obvious that our matrix is still positive-semidefinite.

Multivariate (normal) distributions

- Can be useful to express the correlation structure between 2 variables (even if they are not normal)
  - » Consider a bivariate normal distribution of V_1 and V_2.
  - » Conditional on knowing a realization v_1 of V_1, the value of V_2 is normal with mean

$$E\left[V_2 \middle| v_1\right] = \mu_2 + \rho \frac{v_1 - \mu_1}{\sigma_1}$$

  - » and standard deviation

$$\sigma\left[V_2 \middle| v_1\right] = \sigma_2 \sqrt{1-\rho^2}$$

- Advantages
  - » Many variables can be handled.
  - » A variance-covariance matrix defines the variances of and correlations between variables.
  - » To be internally consistent a variance-covariance matrix must be positive-semidefinite.

Factor models

- Sometimes, it is particularly appreciable to use a factor model to describe the correlation structure between normally distributed variables
  - » With N variables, we only need N parameters instead of $N(N-1)/2$ correlations without any factor model.
- In a 1-factor model (such as the market model), each variable U_i has a standard normal distribution and is represented as a mixture of a common factor F and some specific randomness

common　　　specific

$$\sqrt{\rho^2} + \sqrt{1-\rho^2} = 1$$

$$\begin{cases} U_i = a_i \underbrace{F}_{\text{common}} + \sqrt{1-a_i^2}\, \underbrace{Z_i}_{\text{specific}} \\ \text{where} \\ F, Z_i \quad \sim N(0,1) \\ a_i \quad \text{constant} \in [-1,+1] \\ Z_i \quad \text{uncorrelated with other } Z_i\text{'s and } F \\ \rho(U_i, U_j) \quad = a_i a_j \end{cases}$$

Factor models (2)

- Representation of the multi-factor model

$$U_i = a_{i1}F_1 + a_{i2}F_2 + \ldots a_{iM}F_M + \sqrt{1 - a_{i1}^2 - a_{i2}^2 - \ldots a_{iM}^2}\, Z_i$$

$$\rho\left(U_i, U_j\right) = \sum_{m=1}^{M} a_{im} a_{jm}$$

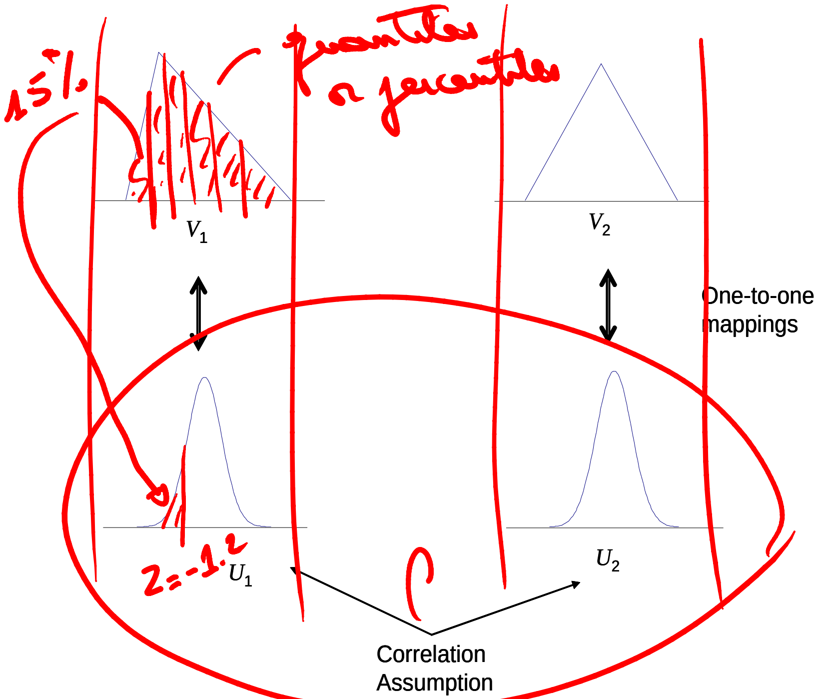
Mapping.

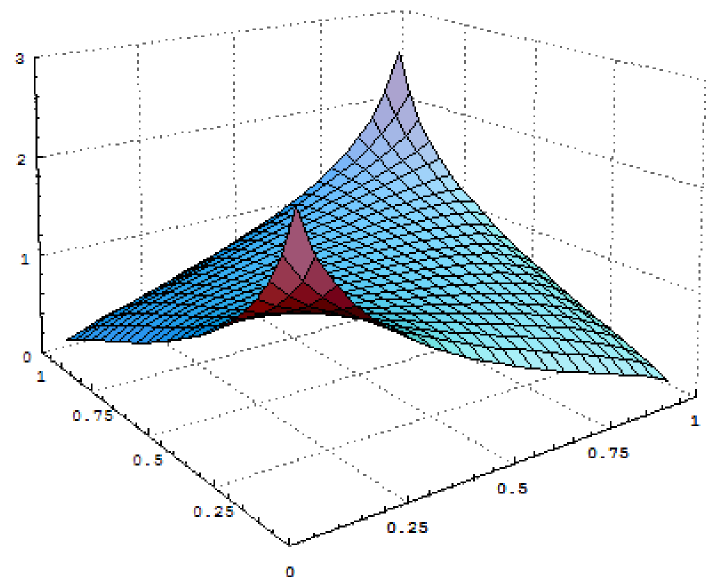
Copulas - Introduction

- Consider two random variables V_1 and V_2, each with its own marginal (or unconditional) distribution
- Suppose now that we want to define a correlation structure between them to obtain a joint distribution
  - If they are normal, then we can assume a bivariate normal joint distribution (many other ways exist)
  - If not normal, unless we work with other well known marginal distributions, there is no natural way to do it.
- We will use copula functions that allow us to map V_1 and V_2 into new variables for which we know some marginal distribution.
  - The mapping is performed on a percentile-to-percentile basis
  - It preserves the original marginal distribution of the original variables
- Depending on which mapping we work, copulas take different names...
  - Gaussian copula: for a normal mapping
  - Student *t*-copula
  - Archimedean copulas (Product, Clayton,...)
  - The Deheuvels or empirical copula

The Gaussian copula

- We transform V_1 and V_2 to new variables U_1 and U_2 that have a standard normal distribution on a “percentile-to-percentile” basis.
- U_1 and U_2 are assumed to have a bivariate normal distribution.





- We can find any joint probability that V_1 and V_2 are less than some specified values, by using the cumulative bivariate distribution with the mapped values U_1 and U_2 and the *copula correlation*.

The Gaussian copula (2)

- Algebraically
  - F_1 and F_2 are the marginal distributions of V_1 and V_2
  - We map $V_1 = v_1$ to $U_1 = u_1$ and $V_2 = v_2$ to $U_2 = u_2$ where

$$F_1(v_1) = N(u_1) \quad \text{and} \quad F_2(v_2) = N(u_2)$$

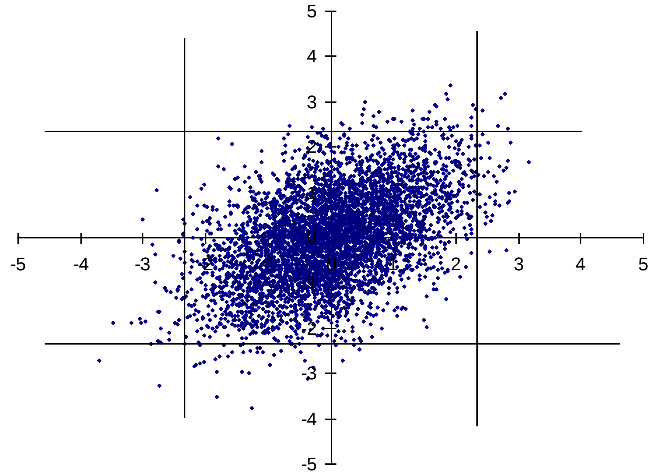
  - This means that

$$u_1 = N^{-1}(F_1(v_1)) \quad \text{and} \quad u_2 = N^{-1}(F_2(v_2))$$

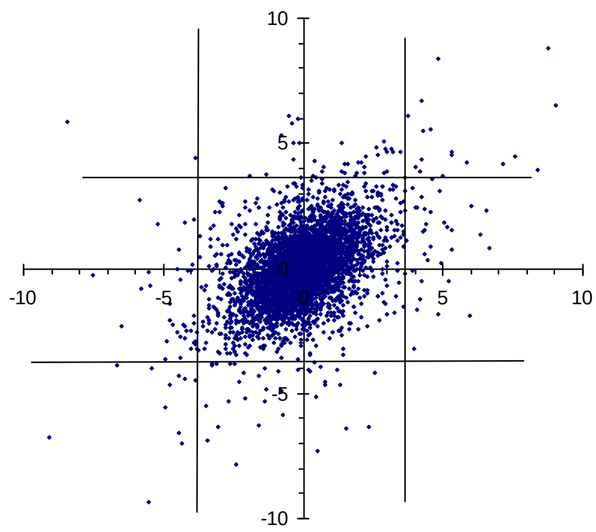
normal variable

Student t-copula

- 5000 random samples from the bivariate normal



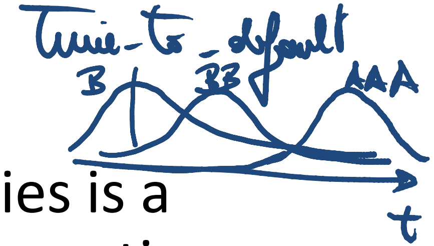
- 5000 random samples from the bivariate Student t



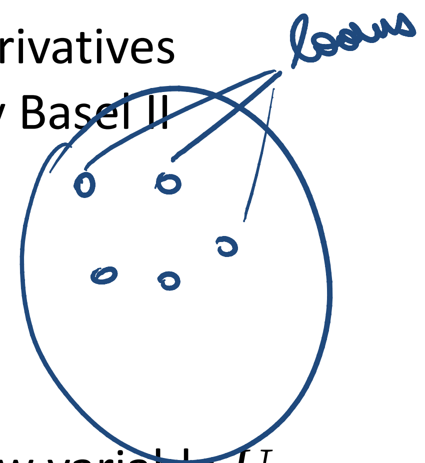
Factor Copula Model

- In a factor copula model the correlation structure between the U's is generated by assuming one or more factors.
  - » Example $U_i = a_i F + \sqrt{1 - a_i^2}\, Z_i$
  - » F and Z_i have standard normal distributions
  - » Other factor copula models can be obtained by choosing F and Z to have other zero-mean unit-variance distributions:
    - ✓ If Z_i is normal
    - ✓ And F is Student t-distributed
    - = Mutivariate Student t-distribution for the U's

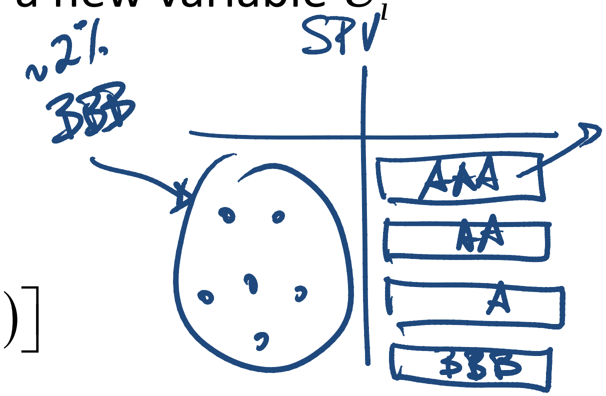
Application to loan portfolios



- The credit default correlation between two companies is a measure of their tendency to default at about the same time
  - » Default correlation is important in risk management when analyzing the benefits of credit risk diversification
  - » It is also important in the valuation of some credit derivatives
  - » We will present a one-factor Gaussian copula used by Basel II



- The Model
  - » Portfolio of N companies
  - » T_i (i = 1..N) is the default hitting time of company i
  - » The cumulative distribution of T_i is Q_i.
  - » We map the time to default for company i, T_i, to a new variable U_i assuming $$U_i = a_i F + \sqrt{1 - a_i^2}\, Z_i$$
  - » The mappings imply $\mathrm{Prob}\left[U_i < U\right] = \mathrm{Prob}\left[T_i < T\right]$
  - » when $N(U) = Q_i(T)$ or $U = N^{-1}\left[Q_i(T)\right]$



Application to loan portfolios (2)

- Conditional on knowing F, we have

we assume that the common factor is knowable

$$\text{Prob}(U_i < U \mid F) = N\left[\frac{U - a_i F}{\sqrt{1-a_i^2}}\right]$$

Hence

frequency of time-of-default that I have as evidence

$$E[U_i/F] = a_i F$$

$$Var[U_i/F] = (1-a_i^2) \cdot 1$$

$$\sigma[U_i/F] = \sqrt{(1-a_i^2)}$$

$$\text{Prob}(T_i < T \mid F) = N\left[\frac{N^{-1}[Q_i(T)] - a_i F}{\sqrt{1-a_i^2}}\right]$$

Assuming the Q's and a's are the same for all companies

$$\text{Prob}(T_i < T \mid F) = N\left[\frac{N^{-1}[Q(T)] - \sqrt{\rho} F}{\sqrt{1-\rho}}\right]$$

base correlation

In a large portfolio the 1-X percentile of F gives the X% worst case percentage of losses in time T

probability of the common factor

$$WCDR(T, X) = N\left[\frac{N^{-1}[Q(T)] + \sqrt{\rho}\, \boxed{N^{-1}(X)}}{\sqrt{1-\rho}}\right]$$

Application to loan portfolios (2)

- Example
  - The average yearly probability of default on every loan of a portfolio of loans is 2%
  - The expected recovery upon default is 40%
  - The copula correlation parameter is estimated at 10%
  - In this case

$$WCDR = N\left[\frac{N^{-1}[2\%]+\sqrt{10\%}\,N^{-1}(99.9\%)}{\sqrt{1-10\%}}\right] = 12.82\%$$

References

- Hull John (2007), **Risk Management for Financial Institutions**, Prentice-Hall.
- Cherubini, Lucciano & Vechiato (2006), **Copula Methods in Finance**, Wiley Finance Series.
- Schönbucher (2003), **Credit Derivative Pricing Models**, Wiley. [Not directly linked to the present chapter but covers nicely the "copula" concept].